Mathematical Probability

by

M. T. WASAN
Professor of Mathematics

QUEEN'S PAPERS IN PURE AND
APPLIED MATHEMATICS — NO. 30

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO
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PREFACE

The origins of mathematical probability are to be found in gambling, a passion of man throughout the ages. This initial approach to probability was intuitive and reasoning was based on a relative frequency approach. In 1930 A. Kolmogorov laid the foundation of mathematical probability by basing it on abstract measure theory, and by considering probability to be an abstract measure with total measure 1. Von Mises immediately criticized this approach, and to this day a cold war continues between the supporters of Radon measures and those of abstract measures. Probability theory is embroiled in this controversy, and as an example we cite the recent paper of L. Schwartz on "Radon measures on Souslin spaces" in which the drawbacks in the foundations of abstract measure theory, and hence of probability theory, are pointed out.

In spite of the fact that there is still disagreement on the foundations of probability theory, it has had a profound influence on almost every branch of mathematics and every scientific discipline. One reason for this is that probability, which is an additive set function, may

conveniently be transformed to a distribution function, which is a point function. The theory of point functions, more commonly known as real variable analysis, is very extensive, powerful, and relatively well-known. In order to tap the rich resources of real variable analysis, and apply them to the theory of probability, this monograph will concentrate on distribution functions.

This manuscript grew out of my lectures to senior students at Queen's University. Their interests varied and may be classified in four categories:

1) Students who studied probability as part of their graduate programs.
2) Undergraduate students with a major in statistics.
3) Students with a main interest in applied mathematics.
4) Students who intended to take up careers and were interested in probability as a mathematical tool.

We tried almost every book on the market for this course, and found that none was appreciated by the students. Thus, my students encouraged me to prepare these lecture notes, and, in fact, helped me in every way. I am very grateful to them. In particular, I thank Miss C. Skinner and Mr. K. Li who

helped me in preparation of the manuscript in many ways. I also thank Mr. S. Quah who helped in preparation of Chapter 9. I am also grateful to Professors A. Basu, J. Higginson and T. Stroud for their comments.

We do not claim that there is any original material in these notes but we would like to add a few comments to the choice of the material.

In chapter 1 we discuss probability as an additive set function

In chapters 2 to 8 we discuss methods of finding distribution functions of the Baire functions of random variables and methods of finding the limiting distribution of sequences of random variables. In chapter 9 the law of the iterated logarithm is considered. This should be of interest to the student of pure mathematics in that this law may be used to solve a problem in number theory which was posed by E. Borel, and it should also be of interest to statistics students since it has applications to sequential analysis. It is possible to consider real variable analysis to be a particular case of the analysis of a random variable (i.e. probability

theory). For this reason our students were curious about the calculus of random variables and we included in chapter 10 such a discussion (i.e. the second order random functions). In the final chapter a number of applications are given in order to satisfy the students of categories three and four who are interested mainly in the solution of real life ptoblems.

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August, 1971

M.T. Wasan

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REFERENCES

We suggest the following references for additional reading:

1. ANDERSON, T.W. An Introduction to Multivariate Statistical Analysis. Wiley, New York (1958).

2. ANSCOMBE, F.J. Large sample theory of sequential estimation. Proc. Cambridge Phil. Soc. 48, 600-607 (1952).

3. BAXTER, G. An analogue of the law of the iterated logarithm. Proc. Amer. Math. Soc. 6, 177-181 (1955).

4. BHARUCHA-REID, A.T. Elements of the Theory of Markov Processes and Their Applications. McGraw-Hill, New York (1960).

5. BLACKWELL, D. & GIRSCHICK, M.A. Theory of Games and Statistical Decisions. Wiley, New York (1954).

6. BLANC-LAPIERRE, A. & FORTET, R. Theory of Random Functions, Vol. 1. Gordon & Breach, New York (1965).

7. BLUM, F.J., HANSON, D.L., & ROSENBLATT, J.I. On the central limit theorem for the sum of a random number of independent random variables, Z. Wahrscheinlichkeits theorie, 1, 389-393 (1963).

8. BRIEMAN, L. Probability. Addison-Wesley, Cambridge, Mass. (1968).

9. CHOVER, J. On Strassen's version of the log log law. Z. Wahrscheinlichkeits theorie 8, 83-90 (1967).

10. CHOVER, J. A law of the iterated logarithm for stable summands. Proc. Amer. Math. Soc. 17, 441-443 (1966).

11. CHOW, Y.S. Y. S. martingales in a σ-finite measure space indexed by directed sets. Trans. Amer. Math. Soc., Vol. 97, 254-285 (1960).

12. CHUNG, K.L. A Course in Probability Theory. Harcourt, Brace & World, New York (1968).

13. CLARK, A.B. & DISNEY, R.L. Probability and Random Processes for Engineers and Scientists. John Wiley, New York (1970).

14. CRAMER, H. Mathematical Methods of Statistics. Princeton University Press (1946).

15. CSÖRGÖ, M. On the strong law of large numbers and the central limit theorem for martingales. Trans. Amer. Math. Soc. 131, 259-275 (1968).

16. ———. Some laws of iterated logarithm for martingales and for strongly multiplicative systems. Submitted for publication.

17. DARLING, D.A. & ROBBINS, N. Iterated logarithm inequalities. Proc. Nat. Acad. Sci. (U.S.A.) 57, 1188-1193 (1967).

18. ———. Inequalities for the sequence of sample means. Proc. Acad. Sci. (U.S.A.) 57, 1577-1589 (1967).

19. ———. Confidence sequences for mean, variance and median. Proc. Nat. Acad. Sci. (U.S.A.) 57, 66-68 (1967).

20. DAVID, F.N. & BARTON, D.E. Combinatorial Chance. Griffin, London (1962).

21. DAWSON, D.A. Equivalence of Markov Processes. Trans. Amer. Math. Soc. 131, 1-31 (1968).

22. DEFARES, J. & SNEDDON, I. An Introduction to the Mathematics of Medicine and Biology. North-Holland (1961).

23. DOOB, J.L. Stochastic Processes. John Wiley, New York (1963).

24. DRAKE, A.W. Fundamentals of Applied Probability Theory. McGraw-Hill, New York (1967).

25. EISER, M.M. Introduction to Mathematical Probability Theory. Prentice Hall, Englewood Cliff, N.J. (1969).

26. EPSTEIN, R.A. The Theory of Gambling and Statistical Logic, Academic Press, New York (1967).

27. ERDŐS, P. On the law of the iterated logarithm. Ann. of Math. 43, 419-436 (1942).

28. ERDŐS, P. & GAL. I.S. On the law of the iterated logarithm, I and II. Proc. Kon. Ned. Akad. Van Wet. 58, 65-84 (1955).

29. ESSEEN, G. Fourier analysis of distribution functions. Acta Math. 77, 1-125 (1945).

30. FELLER, W. Generalization of a probability limit theorem of Cramer. Trans. Amer. Math. Soc. 54, 361-372 (1943).

31. FELLER, W. The general form of the so-called law of iterated logarithm. Trans. Amer. Math. Soc. 54, 373-402 (1943).

32. ———. The fundamental limit theorems in probability. Bull. Amer. Math. Soc. 51, 800-832 (1945).

33. ———. The law of the iterated logarithm for identically distributed random variables. Ann. of Math. 47, 631-638 (1946).

34. ———. A limit theorem for random variables with infinite moments. Amer. J. Math. 68, 257-262 (1946).

35. ———. An Introduction to Probability Theory and Its Applications, Vol. I and II. Wiley, New York (1957, 1966).

36. FISZ, M. Probability Theory and Mathematical Statistics. Wiley, New York (1963).

37. FREEDMAN, D.A. A remark on the law of the iterated logarithm. Ann. Math. Stat. 38, 598-600 (1967).

38. FRY, T.C. Probability and Its Engineering Uses. Van Nostrand, Princeton, N.J., 2nd edition (1928).

39. GAPOSHKIN, V.F. The law of the iterated logarithm for Cesaro's and Abel's methods of summation. Theory of Prob. & Appl. 10, 411-421 (1965).

40. GNEDENKO, B.V. Sur la distribution limit du terme maximum d'une serié aleatoiré. Ann. of Math. Vol. 44, 423-453 (1943).

41. ———. The Theory of Probability. English Translation Chelsea, New York (1962).

42. GNEDENKO, B.V. & KOLMOGOROV, A.N. Limit Distribution for Sums of Independent Random Variables. Addison-Wesley, Cambridge, Mass., (1954).

43. HAJEK, J. Asymptotically most powerful rank-order tests. Ann. Math. Stat. Vol. 33, No. 3, 1124-1147 (1962).

44. HALMOS, P. Measure Theory. Van Nostrand, Princeton, N.J. (1950).

45. HARDY, G.H. & LITTLEWOOD, J.E. Some Problems of dio-phantine approximation. Acta Math. 37, 155-239 (1914).

46. HARTMAN, P. & WINTNER, A. On the Law of the iterated logarithm. Amer. J. Math. 63, 169-176 (1941).

47. HAUSDORFF, F. Grundzuge der Mengenlehre. Leipzig (1913).

48. HYMANS, S.H. Probability Theory with Applications to Econometrics and Decision-Making. Prentice-Hall, Englewood Cliffs, N.J. (1967).

49. ITÔ, K. & McKEAN, H.P. Diffusion Processes and their Sample Paths. Academic Press, New York (1965).

50. KADYROV. V. A theorem on the law of the iterated logarithm for inhomogeneous Markov chain. (Russian) Taskent Tekstil Inst. Kaf. Vyss. Math. Sb. Nauch Issled Rabot Scr. Math. Vyp 19, 185-189 (1964).

51. KARLIN, S. A First Course in Stochastic Processes. Academic Press, New York (1966).

52. KEIFER, J. On large deviations of the empirical distribution functions of vector chain variables and a law of the iterated logarithm. Pac. J. Math. 11, 649-660 (1961).

53. KESTEN, H. An iterated logarithm law for local time. Duke Math. J. 32, 447-456 (1965).

54. KHINCHINE, A.I. Uber dyadische Bruche. Math. Z. 18, 109-116 (1923).

55. ———. Sur une théorème général relatif aux probabilitiés dénombrables. C. R. Acad. Sci., Paris 178, 617-619 (1924).

56. ———. Asymptotische Gestze der Wahrscheinlicheits Vechnung. Ergebn. Math. 2, No. 4, Berlin (1933) Chelsea, New York (1948).

57. KOLMOGOROV, A. Das gesetz des dierierten logarithms. Math. Ann. 101, 126-135 (1929).

58. KUCKAROV, Ja H. The law of the iterated logarithm for non-homogeneous Markov Chain. (Russian) Litovak Math. Sb. 5, 575-585 (1965).

59. LAMPERTI, J. Probability. Benjamin, New York (1966).

60. LECAM, L. Locally asymptotically normal families of distributions. Univ. of Calif Pubs. in Stat., No. 3, 37-98

61. LOEVE, M. Probability Theory. Van Nostrand, Princeton, N.J., 3rd edition (1963).

62. LEVY, P. Sur un theoreme de M. Khinchine. Bull. Sci. Math. 55, 145-160 (1931).

63. ———. Theorie de l'addition des Variables Aleatoires. Gauthier-Villars, Paris, 2nd edition (1954).

64. MARCINKIEWICZ, T. & ZYGMUND, A. Rémarque sur la loi du logarithme iteré. Fund. Math. 29, 215-222 (1937).

65. MESSEL, H. The solution of the fluctuation problem in nucleon cascade theory: homogeneous nuclear matter. Proc. Phys. Soc. (London), ser. A, vol. 65, 465-472 (1952).

66. ———. The development of a nucleon cascade. J. Wilson (ed.). Progress in Cosmic Ray Physics, Vol.2. pp.135-216, North-Holland, Amsterdam (1954).

67. MEYER, P.L. Introductory Probability and Statistical Applications. Addison-Wesley, Reading, Mass. (1965).

68. MILLER, R.G. Simultaneous Statistical Inference. McGraw-Hill, New York (1966).

69. MOGYORODI, J. A central limit theorem for the sum of a random number of independent random variables. Public. Math. Inst. Hungar. Acad. Sci. 7A, 409-424 (1963).

70. MORAN, P.A.P. Introduction to Probability Theory. Oxford (1968).

71. MOOD, A.M. Introduction to the Theory of Statistics. McGraw-Hill, New York (1950).

72. MUNROE, M.E. Theory of Probability. McGraw-Hill, New York (1951).

73. PARZEN, E. Modern Probability Theory and Its Applications. Wiley, New York (1960).

74. PETROV, V.V. On the law of the iterated logarithm. Vestnik Leningrad Univ. Ser. Math. Mekh Astronom. (Russian) 21, 63-67 (1966).

75. ———. On a relation between an estimate of the remainder in the central limit theorem and the law of the iterated logarithm. Theory of Prob & Appl. 11, 454-458 (1966).

76. PETROVSKY, I. Zur ersten randwert aufgabe der Warmeleitungsgleichung. Comp. Math. 1, 383-419 (1935).

77. PFEIFFER, P.E. Concepts of Probability Theory. McGraw-Hill, New York (1965).

78. PHILLIP. W. Ein zentraler grenzwertsatz mit anwendungen auf die zahlen theorie. Z. Währscheinlichkeitstheorie 8, 185-203 (1967).

79. ———. Das gesetz vom iterierten logarithmes fur stark mischende stationaire prozesse. Z. Wahrscheinlichkeits-theorie 8, 204-209 (1967).

80. PHILLIP, W. & STACKELBERG, O.P. Zwei grenzweitsatze für kettenbrüche. To appear.

81. PIELOU, E.C. An Introduction to Mathematical Ecology Wiley-Interscience, New York (1969).

82. PRABHU, N.U. Stochastic Processes. MacMillan, New York (1965).

83. RENYI, A. On the central limit theorem for the sum of a random number of independent random variables. Acta Math. Acad. Sci. Hungar. 11, 97-102 (1960).

84. ———. A general method for proving probability theorems and some applications. (Hungarian) Magy. Tud. Akad. III. Oszt. kozl. 11, 79-105 (1961).

85. ———. A simple proof of a theorem of Borel and of the law of the iterated logarithm. Math. Tidsskr B. 41-48 (1948).

86. REVES, Z. The Laws of Large Numbers. Acad. Press, New York (1968).

87. ROBBINS, H & SIEGMUND, D. Iterated logarithm inequalities and related statistical Procedures. Tech. Report No. 27, Dept. of Statistics, Stanford University (1967).

88. SAATY, T.L. Mathematical Methods of Operations Research. McGraw-Hill, New York (1959).

89. SALEM, R. & ZYGMUND, A. La loi du logarithm iterer pour les series trigonometriques lacunaries. Bull. des. Sci. Math. 74, 209-224 (1950).

90. SARHAN, A.E. & GREENBERG, B.G. Contributions to Order Statistics. Wiley, New York (1962).

91. SKOROKHOD, A.V. Studies in the Theory of Random Processes. Addison-Wesley, Mass. (1965).

92. SONNTAG, R.E. & VAN WHYLEN, G.J. Fundamentals of Statistical Thermodynamics. Wiley, New York (1966).

93. SPITZER, F. Principles of Random Walk. Van Nostrand, Princeton (1964).

94. STÄCKELBERG, O.P. On the law of the iterated logarithm I and II. Proc. Kon. Ned. Akad. Van Wet. 67, 48-67 (1964).

95. ———. On the law of the iterated logarithm for continued fractions. Duke Math. J. 33, 801-820 (1966).

96. STEINHAUS, H. Les probabilities denombrobles et leur rapport a la theorie de la measure. Fund. Math. 4, 286-310 (1922).

97. STRASSEN, V. An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeits theorie 3, 211-226 (1964).

98. ———. A converse to the law of the iterated logarithm. Z. Wahrscheinlichkeits theorie 4, 265-268 (1966).

99. ———. Almost sure behaviour of sums of independent random variables and martingales. Proc. Fifth Berkeley Symposium on Math. Stat. & Prob. Vol. 2, Part I, 315-343 (1967).

100. SVESHORIKOV, A.A. Problems in Probability Theory, Mathematical Statistics, and Theory of Random Functions. Saunders, Philadelphia (1968).

101. Symposium on Stochastic Processes. Journal of the Royal Stat. Society, Series B. Vol. XI, No. 2 (1949).

102. TAYLOR, A.E. General Theory of Functions and Integration. Blaisdell, New York (1965).

103. THOMPSON, W.A., Jr. Applied Probability. Holt, Rinehart, and Winston, U.S.A. (1969).

104. TUCKER, H. A Graduate Course in Probability. Academic Press, New York (1967).

105. USPENSKY, J.V. Introduction to Mathematical Probability. McGraw-Hill, New York (1937).

106. WAERDEN, B.L. van der. Mathematical Statistics. Springer-Verlag, Berlin (1969).

107. WASAN, M.T. Parametric Estimation. McGraw-Hill (1970).

108. ———. Stochastic Approximation. Cambridge Tract in in Mathematics & Mathematical Physics 58 (1969).

109. WEISS, M. On the law of the iterated logarithm. J. Math. Mech. 18, 121-132 (1959).

110. ———. The law of the iterated logarithm for lacunary trigonometric series. Trans. Amer. Math. Soc. 91,

111. ———. On the law of the iterated logarithm for uniformly bounded orthonormal systems. Trans. Amer. Soc. 92, 531-553 (1959).